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EXPONENTIAL GROWTH FOR A FRACTIONAL DIFFERENTIAL EQUATION

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ABSTRACT. This paper concerns the exponential growth of solutions for a fractional differential equation with a fractional damping of order between 0 and 1 in the presence of a source of polynomial type.

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KEYWORDS AND PHRASES. Exponential growth, fractional damping, functional energy .

1. INTRODUCTION

The aim in this paper is to extend a previous work by Tatar [16] where an exponential growth for solutions of a wave equation with fractional damping has been established. This result is obtained by introducing a new functional and using an argument due to Georgiev and Todorova [1] together with some appropriate estimations.

We are interested by the following integro-differential problem

$$u_{tt} + \partial_t^{1+\alpha} u - \Delta u - \gamma \Delta u_t = |u|^{p-1}u, \qquad x \in \Omega,\ t > 0 \tag{1}$$

with boundary conditions

$$u(x,t) = 0, \qquad x \in \partial\Omega,\ t > 0 \tag{2}$$

and initial data

$$u(x,0) = u_0(x),\ \ u_t(x,0) = u_1(x), \qquad x \in \Omega \tag{3}$$

where Ω is a bounded domain of ^{N} $(N \geq 1)$ with a smooth boundary $\partial\Omega$. The functions $u_0(x)$ and $u_1(x)$ are given. The constants p, α and γ are such that $p > 1, -1 < \alpha < 1$ and $\gamma \geq 0$. The notation $\partial_t^{1+\alpha}$ denotes the fractional derivative of order $1+\alpha$ in the Caputo sens (see[12]) defined by

$$\partial_t^{1+\alpha} w(t) = I^{-\alpha} \frac{d}{dt} w(t) \quad \text{for} \ -1 < \alpha < 1 \tag{4}$$

and

$$\partial_t^{1+\alpha} w(t) = I^{1-\alpha} \frac{d^2}{dt^2} w(t) \quad \text{for} \ 0 < \alpha < 1, \tag{5}$$

where $I^{\beta}, \beta > 0$ is the fractional integral

$$I^{\beta} w(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} w(s) ds. \tag{6}$$

For more on fractional integrals and derivatives see also [4, 11, 12, 13].

The problem (1)-(3) was first studied for $\alpha = \frac{1}{2}$ and $\gamma = 0$ by Lokshin [8] and Lokshin and Rok in [9]. Then it has been discuted by Matignon et al. [10] (see also [6] for the existence result).

The equation (1) where $\gamma = 0$ and $\alpha = -1$ has been extensively studied by many authors (see [2, 3, 5, 13, 15]). The authors proved the blow-up of solutions in finite time for sufficiently large initial data.

This paper is a continuation of earlier works discussed by M. Kirane and N.-E. Tatar [7] and N.-E. Tatar [16].

Our paper is organized as follows: In the next section we present some definitions and materials needed in our proofs. Section 3 is devoted to the statement of some results and proof of the exponential growth of solutions.

2. Preliminaries

Let us define the classical functional energy associated to the problem (1)-(3) by

$$E(t) = \int_\Omega \left\{ \frac{1}{2}|u_t|^2 + \frac{1}{2}|\nabla u|^2 - \frac{1}{p+1}|u|^{p+1} \right\} dx. \tag{7}$$

We multiply (1) by u_t and integrate over Ω, we obtain

$$\frac{dE(t)}{dt} = -\frac{1}{\Gamma(-\alpha)} \int_\Omega u_t \int_0^t (t-s)^{-(\alpha+1)} u_t(s)\,ds\,dx - \gamma \int_\Omega |\nabla u_t|^2 dx. \tag{8}$$

Observe that $\frac{dE(t)}{dt}$ is of an undefined sign and the decreasing of the energy is not gauaranteed.

Now, we define the modified functional energy by

$$E_{\epsilon,\gamma}(t) = E(t) - \epsilon \int_\Omega \left\{ u_t u + \frac{\gamma}{2}|\nabla u|^2 \right\} dx, \tag{9}$$

for some $0 < \epsilon < 1$ and $\gamma \geq 0$. If we multiply (1) by $(u_t - \epsilon u)$ and integrate over Ω we get

$$\begin{aligned}
&\frac{d}{dt} \int_\Omega \left\{ \frac{1}{2}|u_t|^2 + \frac{1}{2}(1-\epsilon\gamma)|\nabla u|^2 - \frac{1}{p+1}|u|^{p+1} - \epsilon u_t u \right\} dx \\
&= -\frac{1}{\Gamma(-\alpha)} \int_\Omega u_t \int_0^t (t-s)^{-(\alpha+1)} u_t(s)\,ds\,dx - \gamma \int_\Omega |\nabla u_t|^2 dx - \epsilon \int_\Omega |u_t|^2 dx \\
&+ \frac{\epsilon}{\Gamma(-\alpha)} \int_\Omega u \int_0^t (t-s)^{-(\alpha+1)} u_t(s)\,ds\,dx + \epsilon \int_\Omega |\nabla u|^2 - \epsilon \int_\Omega |u|^{p+1} dx.
\end{aligned}$$

Using definition (9) we can write

$$\begin{aligned}
\frac{dE_{\epsilon,\gamma}(t)}{dt} &= -\frac{1}{\Gamma(-\alpha)} \int_\Omega u_t \int_0^t (t-s)^{-(\alpha+1)} u_t(s)\,ds\,dx - \gamma \int_\Omega |\nabla u_t|^2 dx \\
&\quad - \epsilon \int_\Omega |u_t|^2 dx + \frac{\epsilon}{\Gamma(-\alpha)} \int_\Omega u \int_0^t (t-s)^{-(\alpha+1)} u_t(s)\,ds\,dx \\
&\quad + \epsilon \int_\Omega |\nabla u|^2 - \epsilon \int_\Omega |u|^{p+1} dx.
\end{aligned} \tag{10}$$

Next, for $t \geq 0$ we introduce the auxiliary functional

$$H(t) = e^{-\sigma\epsilon t}E_{\epsilon,\gamma}(t) + \mu F(t) \tag{11}$$

where

$$F(t) = \int_0^t \int_\Omega G(t-s)e^{-\sigma\epsilon s}u_s^2 dxds \tag{12}$$

with

$$G(t) = e^{\beta t}\int_t^{+\infty} e^{-\beta s}s^{-(\alpha+1)}ds. \tag{13}$$

Here, $\mu > 0$, $\beta > 0$ and $\sigma > 0$ are three positive constants that will be precised below.

3. Exponential growth of the solution

Theorem 3.1. *Let $u(x,t)$ be a regular solution of (1)-(2) with $-1 < \alpha < 0$ and $p > 1$. Suppose that the initial data $E_{\epsilon,\gamma}(0) < 0$, then $u(x,t)$ grows up exponentially in the L^{p+1}-norm.*

Proof. If we apply the following formula

$$\frac{d}{dt}\int_{a(t)}^{b(t)} \phi(t,z)dz = b'(t)\phi(t,b(t)) - a'(t)\phi(t,a(t)) + \int_{a(t)}^{b(t)} \frac{d\phi}{dt}(t,z)dz,$$

then the differentiation of $F(t)$ with respect to t yields

$$\begin{aligned}\frac{dF(t)}{dt} &= \int_\Omega G(0)e^{-\sigma\epsilon t}|u_t|^2dx - \int_0^t\int_\Omega (t-s)^{-(2\alpha+3)}e^{-\sigma\epsilon s}|u_t|^2dxds \\ &+ \beta\int_0^t\int_\Omega e^{\beta(t-s)}\int_{t-s}^{+\infty} e^{-\beta z}z^{-(2\alpha+3)}e^{-\sigma\epsilon s}|u_t|^2dxds. \end{aligned}\tag{14}$$

Since

$$G(0) = \int_0^{+\infty} e^{-\beta s}s^{-(2\alpha+3)}ds = \beta^{2(\alpha+1)}\Gamma(2\alpha+4),$$

the relation (14) gives

$$\begin{aligned}\frac{dF(t)}{dt} &= \beta^{2(\alpha+1)}\Gamma(2\alpha+4)e^{-\sigma\epsilon t}\int_\Omega |u_t|^2dx \\ &- \int_0^t\int_\Omega (t-s)^{-(2\alpha+3)}e^{-\sigma\epsilon s}|u_t|^2dxds + \beta F(t).\end{aligned}\tag{15}$$

Now, to calculate $\frac{dH(t)}{dt}$, we differentiate (11) with respect to t. So

$$\frac{dH(t)}{dt} = -\sigma\epsilon e^{-\sigma\epsilon t}E_{\epsilon,\gamma}(t) + e^{-\sigma\epsilon t}E'_{\epsilon,\gamma}(t) + \mu F'(t). \tag{16}$$

Taking into account the definitions (9), (10) and (14), the relation (16) becomes

$$\begin{aligned}\frac{dH(t)}{dt} =& -\left(\frac{\sigma\epsilon}{2}+\epsilon-\mu\beta^{2(\alpha+1)}\Gamma(2\alpha+4)\right)e^{-\sigma\epsilon t}\int_\Omega |u_t|^2dx\\ &-\left(\frac{\sigma\epsilon}{2}-\frac{\sigma\epsilon^2\gamma}{2}-\epsilon\right)e^{-\sigma\epsilon t}\int_\Omega|\nabla u|^2dx+\sigma\epsilon^2e^{-\sigma\epsilon t}\int_\Omega u_tudx\\ &-\left(\epsilon-\frac{\sigma\epsilon}{p+1}\right)e^{-\sigma\epsilon t}\int_\Omega|u|^{p+1}dx-\gamma e^{-\sigma\epsilon t}\int_\Omega|\nabla u_t|^2dx\\ &-\mu\int_0^t\int_\Omega(t-s)^{-(2\alpha+3)}e^{-\sigma\epsilon s}|u_t|^2dxds\\ &+\frac{\epsilon e^{-\sigma\epsilon t}}{\Gamma(-\alpha)}\int_\Omega u\int_0^t(t-s)^{-(\alpha+1)}u_t(s)dsdx\\ &-\frac{\epsilon e^{-\sigma\epsilon t}}{\Gamma(-\alpha)}\int_\Omega u_t\int_0^t(t-s)^{-(\alpha+1)}u_t(s)dsdx+\mu F(t).\end{aligned}\tag{17}$$

The Young inequality and the Poincare inequality give

$$\int_\Omega u_tudx\le\frac{1}{4\epsilon}\int_\Omega|u_t|^2dx+\epsilon C_p\int_\Omega|\nabla u|^2dx\tag{18}$$

where C_p is the Poincare constant. In the first time we can write

$$\begin{aligned}&e^{-\sigma\epsilon t}\int_\Omega u_t\int_0^t(t-s)^{-(\alpha+1)}u_t(s)dsdx\\ &=e^{\frac{-\sigma\epsilon t}{2}}\int_\Omega u_t\int_0^t(t-s)^{-(\alpha+1)}e^{-\frac{\sigma\epsilon}{2}(t-s)}e^{-\frac{\sigma\epsilon s}{2}}u_t(s)dsdx.\end{aligned}$$

Then, the Young inequality yields

$$\begin{aligned}&e^{-\sigma\epsilon t}\int_\Omega u_t\int_0^t(t-s)^{-(\alpha+1)}u_t(s)dsdx\\ &\le\frac{\epsilon\Gamma(-\alpha)}{2}e^{-\sigma\epsilon t}\int_\Omega|u_t|^2dx\\ &+\frac{1}{2\epsilon\Gamma(-\alpha)}\int_\Omega\left(\int_0^t(t-s)^{-(\alpha+1)}e^{-\frac{\sigma\epsilon}{2}(t-s)}e^{-\frac{\sigma\epsilon s}{2}}u_t(s)ds\right)^2dx.\end{aligned}$$

Using the Hölder inequality with the decomposition $\alpha+1=-\frac{1}{2}+(\alpha+\frac{3}{2})$, we obtain

$$\left(\int_0^t(t-s)^{-(\alpha+1)}e^{-\frac{\sigma\epsilon}{2}(t-s)}e^{-\frac{\sigma\epsilon s}{2}}u_t(s)ds\right)^2\le\frac{1}{(\sigma\epsilon)^2}\int_0^t(t-s)^{-(2\alpha+3)}e^{-\sigma\epsilon s}|u_t|^2ds,$$

finally, we arrive at

$$\begin{aligned}&e^{-\sigma\epsilon t}\int_\Omega u_t\int_0^t(t-s)^{-(\alpha+1)}u_t(s)dsdx\\ &\le\frac{\epsilon\Gamma(-\alpha)}{2}e^{-\sigma\epsilon t}\int_\Omega|u_t|^2dx\\ &+\frac{1}{2\Gamma(-\alpha)\sigma^2\epsilon^3}\int_\Omega\int_0^t(t-s)^{-(2\alpha+3)}e^{-\sigma\epsilon s}|u_t|^2dsdx.\end{aligned}\tag{19}$$

Similarly, we have

$$
\begin{aligned}
& e^{-\sigma\epsilon t}\int_\Omega u\int_0^t (t-s)^{-(\alpha+1)}u_t(s)dsdx \\
& \leq \delta C_p e^{-\sigma\epsilon t}\int_\Omega |\nabla u|^2 dx \\
& + \frac{1}{4\delta\sigma^2\epsilon^2}\int_\Omega\int_0^t (t-s)^{-(2\alpha+3)}e^{-\sigma\epsilon s}|u_t|^2 dsdx. \quad (\delta>0).
\end{aligned} \tag{20}
$$

By substitution of (18)-(20) in (17), we get

$$
\begin{aligned}
\frac{dH(t)}{dt} \leq & -\left[\frac{\sigma\epsilon}{2}+\epsilon-\mu\beta^{2(\alpha+1)}\Gamma(2\alpha+4)-\frac{\sigma\epsilon}{4}-\frac{\epsilon}{2}\right]e^{-\sigma\epsilon t}\int_\Omega |u_t|^2 dx \\
& -\epsilon\left[\frac{\epsilon}{2}-\left(1+\frac{\sigma\epsilon\gamma}{2}+\sigma\epsilon^2 C_p+\frac{\delta C_p}{\Gamma(-\alpha)}\right)\right]e^{-\sigma\epsilon t}\int_\Omega|\nabla u|^2 dx \\
& -\left[\mu-\frac{1}{4\sigma^2\epsilon^2\Gamma(-\alpha)}\left(\frac{2}{\epsilon\Gamma(-\alpha)}+\frac{\epsilon}{\delta}\right)\right]\int_0^t\int_\Omega (t-s)^{-(2\alpha+3)}e^{-\sigma\epsilon s}|u_t|^2 dsdx \\
& -\epsilon\left(1-\frac{\sigma}{p+1}\right)e^{-\sigma\epsilon t}\int_\Omega |u|^{p+1}dx-\gamma e^{-\sigma\epsilon t}\int_\Omega|\nabla u_t|^2 dx+\mu\beta F(t).
\end{aligned} \tag{21}
$$

Adding and subtracting $\sigma\epsilon H(t)$ in the right hand side of (21) after application (18) to the term $\int_\Omega u_t u dx$, we obtain:

$$
\begin{aligned}
\frac{dH(t)}{dt} \leq & \ \sigma\epsilon H(t)-\frac{1}{2}\left[\sigma\epsilon+\epsilon-2\mu\beta^{2(\alpha+1)}\Gamma(2\alpha+4)\right]e^{-\sigma\epsilon t}\int_\Omega |u_t|^2 dx \\
& -\epsilon\left[\sigma-\left(1+\sigma\epsilon\gamma+2\sigma\epsilon^2 C_p+\frac{\delta C_p}{\Gamma(-\alpha)}\right)\right]e^{-\sigma\epsilon t}\int_\Omega|\nabla u|^2 dx \\
& -\left[\mu-\frac{1}{4\sigma^2\epsilon^2\Gamma(-\alpha)}\left(\frac{2}{\epsilon\Gamma(-\alpha)}+\frac{\epsilon}{\delta}\right)\right]\int_0^t\int_\Omega (t-s)^{-(2\alpha+3)}e^{-\sigma\epsilon s}|u_t|^2 dxds \\
& -\epsilon\left(1-\frac{2\sigma}{p+1}\right)e^{-\sigma\epsilon t}\int_\Omega |u|^{p+1}dx-\gamma e^{-\sigma\epsilon t}\int_\Omega|\nabla u_t|^2 dx+\mu(\beta-\sigma\epsilon)F(t).
\end{aligned} \tag{22}
$$

To simplify the calculations we choose $\delta=\frac{(p-1\Gamma(-\alpha))}{4C_p}$. Then inequality (22) reduces to

$$
\begin{aligned}
\frac{dH(t)}{dt} \leq & \ \sigma\epsilon H(t)-\frac{1}{2}\left[\sigma\epsilon+\epsilon-2\mu\beta^{2(\alpha+1)}\Gamma(2\alpha+4)\right]e^{-\sigma\epsilon t}\int_\Omega |u_t|^2 dx \\
& -\epsilon\left[\sigma-\left(\sigma\epsilon\gamma+2\sigma\epsilon^2 C_p+\frac{p+3}{4}\right)\right]e^{-\sigma\epsilon t}\int_\Omega|\nabla u|^2 dx \\
& -\left[\mu-\frac{1}{2\sigma^2\epsilon^2\Gamma^2(-\alpha)}\left(\frac{1}{\epsilon}+\frac{2\epsilon C_p}{p-1}\right)\right]\int_0^t\int_\Omega (t-s)^{-(2\alpha+3)}e^{-\sigma\epsilon s}|u_t|^2 dxds \\
& -\epsilon\left(1-\frac{2\sigma}{p+1}\right)e^{-\sigma\epsilon t}\int_\Omega |u|^{p+1}dx-\gamma e^{-\sigma\epsilon t}\int_\Omega|\nabla u_t|^2 dx+\mu(\beta-\sigma\epsilon)F(t).
\end{aligned} \tag{23}
$$

Now, note that the coefficient of $\int_\Omega |\nabla u_t|^2 dx$ in (23) is negative.
Next, if we choose (with simple conditions)

$$\epsilon < \min\left\{1, \frac{1}{\gamma + C_p}, \frac{-\gamma + \sqrt{\gamma^2 + 8C_p}}{4C_p}, \frac{-\gamma(p+1) + \sqrt{\gamma^2(p+1)^2 + 4C_p(p^2-1)}}{4C_p(p+1)}\right\},$$

then, it is possible to select σ such that

$$\frac{p+3}{4(1-\gamma\epsilon - 2\epsilon^2 C_p)} < \sigma < \frac{p+1}{2},$$

which guarantees the negativity of the coefficients of $\int_\Omega |\nabla u|^2 dx$ and $\int_\Omega |u|^{p+1} dx$.

We assume that μ is large enough, as

$$\mu \geq \frac{1}{2\sigma^2\epsilon^2\Gamma^2(-\alpha)}\left(\frac{1}{\epsilon} + \frac{2\epsilon C_p}{p-1}\right)$$

and

$$\beta \leq \min\left\{\sigma\epsilon, \left[\frac{\epsilon}{2\mu\Gamma(2\alpha+4)}\right]^{\frac{1}{2(\alpha+1)}}\right\},$$

the other coefficients in (23) are all negative. This allows us to write (23) as

$$\frac{dH(t)}{dt} \leq \sigma\epsilon H(t) \qquad (t \geq 0). \tag{24}$$

From the hypothesis of the theorem 3.1 we have

$$\begin{aligned} H(0) &= E_{\epsilon,\gamma}(0) \\ &= \int_\Omega \left\{\frac{1}{2}u_1^2 + \frac{1}{2}(1-\epsilon\gamma)|\nabla u_0|^2 - \frac{1}{p+1}|u_0|^{p+1} - \epsilon u_0 u_1\right\} dx < 0. \end{aligned}$$

Using the differential form of the Gronwall inequality we obtain directly from (24) that

$$H(t) \leq H(0)e^{\sigma\epsilon t} \qquad (t \geq 0). \tag{25}$$

On the other hand, from the definition of $H(t)$ we can write

$$\begin{aligned} H(t) \geq &- \frac{e^{-\sigma\epsilon t}}{p+1}\int_\Omega |u|^{p+1} dx + \frac{e^{-\sigma\epsilon t}}{2}\int_\Omega |u_t|^2 dx + \frac{1}{2}(1-\gamma)e^{-\sigma\epsilon t}\int_\Omega |\nabla u|^2 dx \\ &- \epsilon e^{-\sigma\epsilon t}\int_\Omega u_t u\, dx. \end{aligned}$$

Applying (18) (with $\epsilon = \frac{1}{2}$) we get for $t \geq 0$,

$$\begin{aligned} H(t) \geq &- \frac{e^{-\sigma\epsilon t}}{p+1}\int_\Omega |u|^{p+1} dx \\ &+ \frac{e^{-\sigma\epsilon t}}{2}\int_\Omega \left[(1-\epsilon)|u_t|^2 + (1-\epsilon\gamma - \epsilon C_p)|\nabla u|^2\right] dx. \end{aligned}$$

By our choice of ϵ, we have $1-\epsilon > 0$ and $1 - \epsilon\gamma - \epsilon C_p > 0$, then it is clear that

$$H(t) \geq -\frac{e^{-\sigma\epsilon t}}{p+1}\int_\Omega |u|^{p+1} dx. \tag{26}$$

Inequalities (25) and (26) lead to

$$H(0)e^{\sigma\epsilon t} \geq -\frac{e^{-\sigma\epsilon t}}{p+1}\int_\Omega |u|^{p+1}dx,$$

which implies that

$$\int_\Omega |u|^{p+1}dx \geq -H(0)(p+1)e^{2\sigma\epsilon t} \qquad (t \geq 0).$$

The proof is now complete. □

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